Inverse Scattering Transform and Space -Time Scattering for the KPI equation

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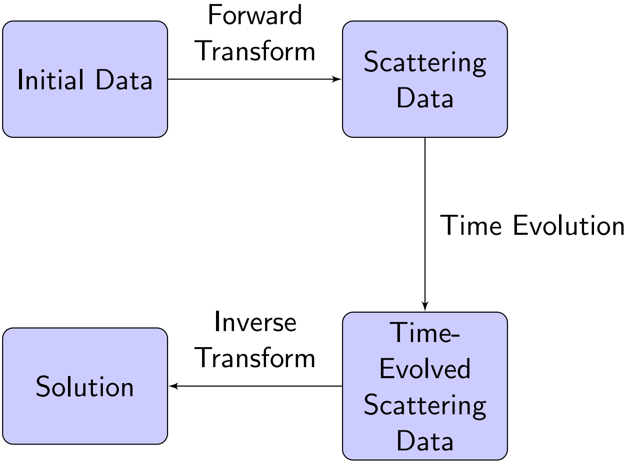
Schematic Description of Solving KPI using IST



Content

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Lax Pair Representation for the KPI Equation

The Cauchy problem for the KPI equation is given by

$$\begin{cases} (u_t + 6uu_x + u_{xxx})_x = 3u_{yy} \\ u(0,x,y) = u(x,y) \end{cases} \quad (1)$$

A Lax pair for the KPI is given by

$$\widehat{L}\psi = i\psi_y + \psi_{xx} + u\psi = 0 \quad (2)$$

and

$$\widehat{M}\psi = \psi_t + 4\psi_{xxx} + 6u\psi_x + 3\psi\left[u_x - i\int_{-\infty}^{x} u_y\,dx'\right] = 0 \quad (3)$$

where the KPI equation is the compatibility condition for (2) and (3), i.e., (1) can be written as Lax's equation for $\widehat{L}$ and $\widehat{M}$,

$$\widehat{L}_t = i[\widehat{L}, \widehat{M}].$$

Integral Equations

Consider an accompanying equation of (2) is given by

$$-i\phi_y + \phi_{xx} + u\phi = 0 \tag{4}$$

Let $\mu = e^{-i(kx+k^2y)}\psi$ and $\nu = e^{i(kx-k^2y)}\phi$. Then, equations (2) and (4) can be written as

$$L\mu \equiv i\mu_y + \mu_{xx} + 2ik\mu_x + u\mu = 0 \tag{5}$$

$$\tilde{L}\nu \equiv -i\nu_y + \nu_{xx} - 2ik\nu_x + u\nu = 0 \tag{6}$$

Taking the Fourier and inverse Fourier transforms of (5) and (6), respectively,

$$i\widehat{\mu}_y - (l^2 + 2kl)\widehat{\mu} = -(2\pi)^{-1}\widehat{u} * \widehat{\mu} \tag{7}$$

$$i\check{\nu}_y + (l^2 + 2kl)\check{\nu} = \check{u} * \check{\nu} \tag{8}$$

Integral Equations (cont.)

Denote the solutions of (7) by

$$\widehat{\mu}^{l}(l,y;k)=2\pi\delta(l)+i(2\pi)^{-1}\int_{-\infty}^{y}e^{-il(l+2k)(y-\eta)}\widehat{u}*\widehat{\mu}^{l}(l,\eta)\;d\eta \qquad (9)$$

$$\widehat{\mu}^{r}(l,y;k)=2\pi\delta(l)-i(2\pi)^{-1}\int_{y}^{+\infty}e^{-il(l+2k)(y-\eta)}\widehat{u}*\widehat{\mu}^{r}(l,\eta)\;d\eta \qquad (10)$$

$$\widehat{\mu}^{\pm}(l,y;k)=2\pi\delta(l)+i(2\pi)^{-1}\int_{\pm\infty\cdot l}^{y}e^{-il(l+2k)(y-\eta)}\widehat{u}*\widehat{\mu}^{\pm}(l,\eta)\;d\eta \qquad (11)$$

Note:

(1) The equations (9) and (10) have unique solutions if $\widehat{u}\in L^1(\mathbb{R}^2)$.

(2) On the other hand, existence of unique solutions to (11) requires $\|\widehat{u}\|_{L^1(\mathbb{R}^2)}<2\pi$ if (11) is solved iteratively.

Scattering Data

Define

$$S(k,k+l) = -i(2\pi)^{-2} \int e^{il(l+2k)\eta} \hat{u} * \hat{\mu}^r(l,\eta;k)\ d\eta \tag{12}$$

$$\tilde{S}(k+l,k) = -i(2\pi)^{-2} \int e^{-il(l+2k)\eta} \check{u} * \check{\nu}^r(l,\eta;k)\ d\eta \tag{13}$$

and

$$T^{\pm}(k,k+l) = -i(2\pi)^{-2} H(\pm l) \int e^{il(l+2k)\eta} \hat{u} * \hat{\mu}^{\pm}(l,\eta;k)\ d\eta \tag{14}$$

$$\tilde{T}^{\pm}(k+l,k) = i(2\pi)^{-2} H(\mp l) \int e^{-il(l+2k)\eta} \check{u} * \check{\nu}^{\pm}(l,\eta;k)\ d\eta \tag{15}$$

$$R^{\pm}(k,k+l) = i(2\pi)^{-2} H(\mp l) \int e^{il(l+2k)\eta} \hat{u} * \hat{\mu}^{\pm}(l,\eta;k)\ d\eta \tag{16}$$

$$\tilde{R}^{\pm}(k+l,k) = -i(2\pi)^{-2} H(\pm l) \int e^{-il(l+2k)\eta} \check{u} * \check{\nu}^{\pm}(l,\eta;k)\ d\eta \tag{17}$$

Scattering Data (cont.)

Triangular factorization of $I + \mathcal{S}$ is given by

$$I + \mathcal{S} = (I \pm \mathcal{R}^{\pm})^{-1}(I \pm \mathcal{T}^{\pm}) \tag{18}$$

It follows that

$$I + \mathcal{F} \equiv (I + \mathcal{T}^{+})(I - \tilde{\mathcal{T}}^{-}) = (I + \mathcal{R}^{+})(I - \tilde{\mathcal{R}}^{-}) \tag{19}$$

relates the analytical solutions $\mu^{\pm}$ through a nonlocal Riemann-Hilbert problem

$$\mu^{+} = (I + \mathcal{F}_{x,y})\mu^{-} \tag{20}$$

where $\mathcal{F}_{x,y}$ denotes the integral operator with the kernel $F(k,l)e^{i(l-k)x - i(l^2-k^2)y}$ for fixed x, y.

Alternatively, (20) can be rewritten as

$$\mu^{\pm} = (I \pm \mathcal{T}^{\pm}_{x,y})\mu^{l} \tag{21}$$

$$\mu^{\pm} = (I \pm \mathcal{R}^{\pm}_{x,y})\mu^{r} \tag{22}$$

Inverse Problem (cont.)

Define the following integral operators for the nonlocal Riemann-Hilbert problem

$$C_{T_{x,y}} \equiv C_+ \mathcal{T}^-_{x,y} + C_- \mathcal{T}^+_{x,y} \quad (23)$$

$$C_{R_{x,y}} \equiv C_+ \mathcal{R}^-_{x,y} + C_- \mathcal{R}^+_{x,y} \quad (24)$$

where $C_\pm : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ are Cauchy integral operators defined by

$$(C_\pm f)(k,l) = \frac{1}{2\pi i} \int \frac{dk'}{k' - k \mp 0i} f(k',l) \quad (25)$$

One can show that $\mu^{l,r}$ are fundamental solutions of the Riemann-Hilbert problems (21) and (22) if and only if they satisfy the following equations, respectively

$$\mu^l = 1 + C_{T_{x,y}} \mu^l \quad (26)$$

$$\mu^r = 1 + C_{R_{x,y}} \mu^r \quad (27)$$

Inverse Problem (cont.)

Let u be an undetermined function of x, y. Using (26),

$$\begin{aligned}[C_{T_{x,y}}, L - u]\mu^l &= [C_{T_{x,y}}, L]\mu^l \\ &= C_{T_{x,y}} L\mu^l - LC_{T_{x,y}}\mu^l \\ &= (C_{T_{x,y}} - I)L\mu^l + u\end{aligned}$$

where the operator L is defined in (5). If we set

$$u = [C_{T_{x,y}}, L - u]\mu^l = [C_{T_{x,y}}, i\partial_y + 2ik\partial_x + \partial_{xx}]\mu^l \tag{28}$$

then, $L\mu^l = 0$ by the injectivity of $I - C_{T_{x,y}}$.

Using (23) and (25), we can write (28) as

$$u(x,y) = \frac{1}{\pi}\frac{\partial}{\partial x}\iint [T^+(k,l) + T^-(k,l)]e^{i(l-k)x - i(l^2-k^2)y}\mu^l(l,x;y)\, dl\, dk \tag{29}$$

Inverse Problem (cont.)

(1) We previously showed that the physical scattering kernel evolves as $S(k, l, t) = S(k, l)e^{4i(k^3 - l^3)t}$.

(2) The triangular factors $T^{\pm}$ and $R^{\pm}$ evolve in the same way.

(3) Thus the KPI equation (1) has a unique solution $u(\cdot, \cdot, t)$ for all real t given by (29) at initial time $t = 0$ which evolves in a manner determined by the evolution of the scattering data.

Space-Time Scattering: Definitions and Notations

Denote the closed unit square $[0,1]x[0,1]$ in $\mathbb{R}^2$ by $\square$.

Write $P \in \square$, $P = (1/q, 1/r)$, $1 \leq q \leq \infty$, $1 \leq r \leq \infty$ with the convention that $1/\infty = 0$.

We use the notation

$$L(P) = L^r(\mathbb{R}; L^q(\mathbb{R}^d)), \quad P = (1/q, 1/r) \in \square.$$

For $P \in \square$ write $P = (x(P), y(P))$ for the coordinates.

Space-Time Scattering: Definitions

Definition 1

A distribution $u \in \mathcal{S}'(\mathbb{R}x\mathbb{R}^d)$ is called a *free wave*, if

$$(i\partial_t + \Delta_x)u = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}x\mathbb{R}^d).$$

Definition 2

Let $P \in \square$. Define $\mathcal{L}_0(P) = \{u \in L(P) \mid u \text{ is a free wave}\}$.

Lemma 3

$\mathcal{L}_0\left(\left(\frac{1}{2}, 0\right)\right) = \{e^{i(t-s)\Delta}\psi \mid s \in \mathbb{R},\ \psi \in L^2(\mathbb{R}^d)\}$.

Definition 4

Let $u \in L(P), P \in T$. We say that $(i\partial_t - H(t))u = 0$ holds in the weak sense, if $< (i\partial_t - H(t)\Psi), u) >= 0$ for all $\Psi \in \mathcal{S}(\mathbb{R}x\mathbb{R}^d)$.

Space-Time Scattering (cont.)

Theorem 5

Let V satisfy Assumption 1 and Assumption 2. Let P be V-admissible.

(i) *Let $u \in L(P)$ satisfy $(i\partial_t - H(t))u = 0$ in the weak sense. Then there exist unique free waves $u_\pm \in \mathcal{L}_0(P)$ such that*

$$u \sim u_\pm \quad at \ \pm\infty.$$

Furthermore, the map $u_- \mapsto u_+$ is given by

$$u_+ = (1 + iG_+^0 V)(1 + iG_-^0 V)^{-1} u_-.$$

(ii) *Let $u_- \in \mathcal{L}_0(P)$. Then $u = (1 - iG_- V)u_- \in L(P)$ solves $(i\partial_t - H(t))u = 0$ in the weak sense, and $u \sim u_-$ at $-\infty$. An analogous result holds in the $+\infty$ case.*

Space-Time Scattering (cont.)

Theorem 6

Let V satisfy Assumption 1 and Assumption 2, and let $P \in T$ be V-admissible. Then the following results hold on $\mathcal{L}_0(P)$:

$$\mathrm{W}_{\pm} = 1 - iG_{\pm}V,$$

$$\mathrm{S} = \mathrm{W}_{+}^{-1}\mathrm{W}_{-}.$$

Preliminaries

(1) The Schrödinger equation

$$i\frac{d}{dt}\psi(t) = (-\Delta + V(t))\psi(t), \quad \psi(s) = \psi_0 \tag{30}$$

has the propagator $U(t,s)$ such that the weak solution of (30) is given by $\psi(t) = U(t,s)\psi_0$.

(2) The family $U(t,s)$ consists of unitary propagators acting on $\mathcal{H} = L^2(\mathbb{R}^d)$ with $U(t,t) = I$ and $U(t,s)U(s,r) = U(t,s)$ for all $t, s, r \in \mathbb{R}$.

(3) Denote the propagator for the free Schrödinger equation by $U_0(t) = e^{it\Delta}$, where the domain $\mathcal{D}(-\Delta) = H^2(\mathbb{R}^d)$.

(4) Denote the Banach space of finite regular measures on $\mathbb{R}^d$ by $\mathcal{M}(\mathbb{R}^d)$.

Existence of Wave Operators

Assumption 1

Let $V(t,x)$ be a real-valued function such that $\widehat{V} \in L^1(\mathbb{R}; \mathcal{M}(\mathbb{R}^d))$.

Theorem 7

Let V satisfy Assumption 1. Then the following results hold:

(i) *For each $s \in \mathbb{R}$ the limits*

$$W_{\pm}(s) = \lim_{t \to \pm\infty} U(s,t) U_0(t-s)$$

exist in operator norm in $\mathcal{B}(L^2(\mathbb{R}^d))$ and are unitary.

(ii) *The operators $W_{\pm}(s)$ extends to bounded operators on $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. Furthermore, $W_{\pm}(s)$ are invertible in $\mathcal{B}(L^p(\mathbb{R}^d))$, and we have*

$$\sup_{s \in \mathbb{R}} \left\| W_{\pm}(s) \right\|_{\mathcal{B}(L^p)} < \infty, \quad \sup_{s \in \mathbb{R}} \left\| W_{\pm}(s)^{-1} \right\|_{\mathcal{B}(L^p)} < \infty.$$

Definitions and Notations

Let $\phi \in L^2(\mathbb{R}^d)$ and $s \in \mathbb{R}$. Define two operators from $L^2(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}; L^2(\mathbb{R}^d))$ as

$$\Gamma_0(s)\phi = U_0(t-s)\phi,$$
$$\Gamma(s)\phi = U(t,s)\phi.$$

For $f \in C_c(\mathbb{R}; L^2(\mathbb{R}^d))$ the adjoints are

$$\Gamma_0(s)^* f = \int_{-\infty}^{\infty} U_0(s-t)f(t)dt,$$
$$\Gamma(s)^* f = \int_{-\infty}^{\infty} U(s,t)f(t)dt.$$

For $f \in C_c(\mathbb{R}; L^2(\mathbb{R}^d))$ define maps with values in $L^\infty(R; L^2(\mathbb{R}^d))$ as

$$(G_\pm^0 f)(t) = \int_{\pm\infty}^{t} U_0(t-s)f(s)ds,$$
$$(G_\pm f)(t) = \int_{\pm\infty}^{t} U(t,s)f(s)ds.$$

Definitions and Notations (cont.)

It follows that for any $s \in \mathbb{R}$,

$$G_-^0 - G_+^0 = \Gamma_0(s)\Gamma_0(s)^*,$$
$$G_- - G_+ = \Gamma(s)\Gamma(s)^*.$$

Define wave operators on $L^\infty(\mathbb{R}; L^2(\mathbb{R}^d))$ as

$$(\mathrm{W}_\pm f)(t) = W_\pm(t)f(t)$$

We have the intertwining relation in $\mathcal{B}(L^2(\mathbb{R}^d))$

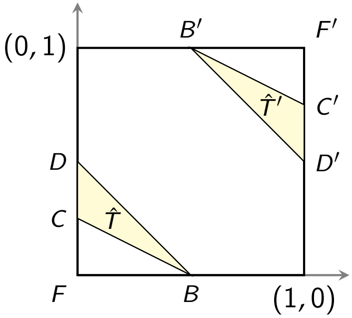
$$U(t,s)W_\pm(s) = W_\pm(t)U_0(t-s), \ t,s \in \mathbb{R}.$$

Using the intertwining relation, we obtain

$$G_+ = \mathrm{W}_+ G_+^0 {\mathrm{W}_+}^{-1},$$
$$G_- = \mathrm{W}_- G_-^0 {\mathrm{W}_-}^{-1},$$
$$\Gamma(s) = \mathrm{W}_+\Gamma_0(s)\mathrm{W}_+(s)^{-1},$$
$$\Gamma(s) = \mathrm{W}_-\Gamma_0(s)\mathrm{W}_-(s)^{-1}$$

Definitions and Notations (cont.)

For $d = 1$,



$$B = \left(\frac{1}{2}, 0\right) \qquad B' = \left(\frac{1}{2}, 1\right) \qquad C = \left(0, \frac{1}{4}\right) \qquad C' = \left(1, \frac{3}{4}\right)$$

$$D = \left(0, \frac{1}{2}\right) \qquad D' = \left(1, \frac{1}{2}\right) \qquad F = (0, 0) \qquad F' = (1, 1)$$

$G^0_\pm$, $G_\pm$ are bounded from $L(Q)$ to $L(P)$

Definition 8

Define the function $\pi : \square \to \mathbb{R}$ by $\pi(P) = x(P) + 2y(P)/d$.

Theorem 9

Assume $P \in T$, $Q \in T'$, and $\pi(Q) - \pi(P) = 2/d$. Then, $G^0_\pm \in \mathcal{B}(L(Q), L(P))$ and $G_\pm \in \mathcal{B}(L(Q), L(P))$.

Lemma 10

$G^0_\pm$ is bounded on $L(\bar{P})$ to $L(P)$ if either

(i) *$P \in [BC[$ and $\bar{P} \in [B'C'[$, or*

(ii) *$P \in T$ and $\bar{P} \in T'$ with*
$x(P) + x(\bar{P}) = 1$, $x(\bar{P}) + 2y(\bar{P})/d - x(P) - 2y(P)/d = 2/d$.

$\Gamma_0(s)$, $\Gamma(s)$ are bounded from $L^p(\mathbb{R}^d)$ to $L(P)$

Theorem 11

Assume $1/2 \leq 1/p \leq d/2(d-1)$ *(for* $d = 1$ *assume* $1/2 \leq 1/p \leq 1$*).*
Let $P \in \widehat{T} \cup [BC[$ *with* $\pi(P) = 1/p$*. Then,*

$$\Gamma_0(s), \Gamma(s) \in \mathcal{B}(L^p(\mathbb{R}^d), L(P)).$$

Let q *be the conjugate of* p *and let* $Q \in \widehat{T}' \cup [B'C'[$ *with* $\pi(Q) = 1/q + 2/d$*. Then*

$$\Gamma_0(s)^*, \Gamma(s)^* \in \mathcal{B}(L(Q), L^q(\mathbb{R}^d)).$$

More Assumptions and Definitions

Assumption 2

Let V be a real-valued function such that $V \in L(R)$ for some $R \in \square$ satisfying $y(R) > 0$ and $\pi(R) = 2/d$.

Definition 12

Let V satisfy Assumption 2 for some R. A pair $P, Q \in \square$ is called V-admissible, if $P \in T$, $Q \in T'$, and $Q = P + R$.

Lemma 13

Let V satisfy Assumption 1 and Assumption 2. Let P, Q be a V-admissible pair. Then the following identities hold in $\mathcal{B}(L(Q), L(P))$

$$G_-^0 - G_- = iG_-^0 V G_- = iG_- V G_-^0,$$

$$G_+^0 - G_+ = iG_+^0 V G_+ = iG_+ V G_+^0$$

Lemma 14

Let V satisfy Assumption 1 and Assumption 2, and let P, Q be a V-admissible pair. Then $1 + iG_\pm^0 V$ is invertible in $\mathcal{B}(L(P))$ with inverse $1 - iG_\pm V$.

Connection between Space-Time Scattering and Inverse Scattering

Consider the Schrödinger equation with a time-dependent potential

$$\begin{cases} i\psi_t + \psi_{xx} - V\psi = 0 \\ \lim\limits_{t\to\pm\infty} |\psi(x,t) - e^{i(kx-k^2t)}| = 0 \end{cases} \tag{31}$$

It follows that a solution $\psi^l(x,t)$ of (31), which is a $L^\infty(\mathbb{R})$-valued function of t, also solves

$$\psi^l(x,t) = e^{ikx-ik^2t} + \int_{-\infty}^{t} e^{i(t-s)\Delta}(iV\psi^l)(s)ds \tag{32}$$

Theorem 15 (Main Theorem 1)

Assume $\widehat{V} \in L^1(\mathbb{R}^2)$. *Let* ψ^l *solve* (32) *and let*

$$\psi^l_+(x,t) = e^{i(kx-k^2t)} + \sum_{n=1}^{\infty} \int e^{i(\xi x - \xi^2)t} S^l_n(\xi,k)\, d\xi \tag{33}$$

where for $n \geq 1$,

$$S^l_n(\xi,k) = \frac{(-i)^n}{(2\pi)^n} \iint \left(\prod_{j=0}^{n-1} e^{it_j(\xi^2_{j-1} - \xi^2_j)} \widehat{V}(t_j, \xi_{j-1} - \xi_j) \right) d^n t\, d^{n-1}\xi \tag{34}$$

where the t *integration goes over* $-\infty < t_{n-1} \leq t_{n-2} \leq \ldots \leq t_0 < \infty$ *and the* ξ *integration goes over* $(\xi_0, \ldots, \xi_{n-2}) \in \mathbb{R}^{n-1}$, *and* $\xi_{-1} = \xi$ *and* $\xi_{n-1} = k$. *Then,*

$$\lim_{t \to +\infty} |\psi^l(x,t) - \psi^l_+(x,t)| = 0$$

i.e., ψ^-_+ *is the outgoing free wave for* ψ^l.

Proof of Theorem 15.

Taking the Fourier transform of (32) in x variable

$$\widehat{\psi}^l(\xi,t) = 2\pi\delta(\xi-k)e^{-ik^2t} + \int_{-\infty}^{t}\int e^{-i(t-s)\xi^2} i\widehat{V}(s,\xi-\eta)\widehat{\psi}^l(\eta,s)\,d\eta\,ds \quad (35)$$

To solve (35) by iteration, let

$$\widehat{\psi}^l(\xi,t) = 2\pi\delta(\xi-k)e^{-i\xi^2t} + \sum_{n=1}^{\infty}\widehat{\psi}^l_n(\xi,t)$$

with

$$\widehat{\psi}^l_n(\xi,t) = -\frac{i}{2\pi}e^{-i\xi^2t}\int_{-\infty}^{t}\int e^{i\xi^2t_1}\widehat{V}(t_0,\xi-\xi_0)\widehat{\psi}^l_{n-1}(\xi_0,t_0)\,d\xi_0\,dt_0$$

for $n \geq 1$.

Proof of Theorem 15 (cont.)

Then, it follows that for $n \geq 1$,

$$\widehat{\psi}_n^l(\xi, t) = e^{-i\xi^2 t} \frac{(-i)^n}{(2\pi)^{n-1}} \iint \left(\prod_{j=0}^{n-1} e^{it_j(\xi_{j-1}^2 - \xi_j^2)} \widehat{V}(t_j, \xi_{j-1} - \xi_j) \right) d^n t \, d^{n-1}\xi \tag{36}$$

where integration goes over $(t_0, \ldots, t_{n-1})$ with $-\infty < t_{n-1} \leq t_{n-2} \leq \ldots \leq t_0 < t$ and $(\xi_0, \ldots \xi_{n-2}) \in \mathbb{R}^{n-1}$ with $\xi_{-1} = \xi$ and $\xi_{n-1} = k$.

Proof of Theorem 15 (cont.)

Taking inverse Fourier transform of (36),

$$\psi^l(x,t) = e^{(ikx-k^2t)} + \sum_{n=1}^{\infty} \int e^{i(\xi x - \xi^2 t)} A_n(\xi,k,t)\, d\xi \tag{37}$$

where

$$A_n(\xi,k,t) = \frac{(-i)^n}{(2\pi)^n} \iint \left(\prod_{j=0}^{n-1} e^{it_j(\xi_{j-1}^2 - \xi_j^2)} \widehat{V}(t_j, \xi_{j-1} - \xi_j) \right) d^n t\, d^{n-1}\xi \tag{38}$$

where integration goes over $(t_0, \ldots, t_{n-1})$ with $-\infty < t_{n-1} \le t_{n-2} \le \ldots \le t_0 < t$ and $(\xi_0, \ldots, \xi_{n-2}) \in \mathbb{R}^{n-1}$ with $\xi_{-1} = \xi$ and $\xi_{n-1} = k$.

Proof of Theorem 15 (cont.)

Note that

$$|A_n(\xi, k, t)| \leq \frac{1}{n!} \left(\left\| \widehat{V} \right\|_{L^1(\mathbb{R}, L^1(\mathbb{R}))} \right)^n$$

So, the series on (37) is absolutely and uniformly convergent.

By (34) and (37),

$$\left| S_n'(\xi, k) - A_n(\xi, k, t) \right| \leq \frac{1}{(n-1)!} \left(\| \widehat{V} \|_{L^1(\mathbb{R}, L^1(\mathbb{R}))} \right)^{n-1} \int_t^{\infty} \int \left| \widehat{V}(s, \xi) \right| d\xi \, ds$$

goes to 0 as $t \to +\infty$. Then by (33) and (38),

$$\lim_{t \to +\infty} |\psi^I(x, t) - \psi_+^I(x, t)| = 0$$

□

Theorem 16 (Main Theorem 2: Explicit Form of Space-Time Scattering)

Let $e^{i(kx-k^2t)}$ be a free wave. Then the action of the space-time scattering operator S *on the free wave is given by*

$$\mathrm{S}\left(e^{i(kx-k^2t)}\right) = e^{i(kx-k^2t)} + \sum_{n=1}^{\infty} \int e^{i(\xi x-\xi^2 t)} S_n'(\xi,k)\,d\xi$$

where S_n' are given by (34).

Proof of Main Theorem 16.

We have the following series expansion for S,

$$S = (I + iG^0_+ V)\left(\sum_{n=0}^{\infty}(-iG^0_- V)^n\right) = I + \sum_{n=1}^{\infty}(iG^0_+ V - iG^0_- V)(-iG^0_- V)^{n-1} \tag{39}$$

Let $g(t) = (-iG^0_- V)^{n-1} f(t)$. Observe that

$$(iG^0_+ V - iG^0_- V)g(t) = -i\int_{-\infty}^{\infty} U_0(t-s)V(s)g(s)\, ds \tag{40}$$

while

$$\begin{aligned}&(-iG^0_- V)^{n-1} f(t)\\ &= (-i)^{n-1}\int_{\{-\infty<t_{n-1}\leq\ldots\leq t_1\leq t\}}\left(\prod_{j=1}^{n-1} U_0(t_{j-1}-t_j)V(t_j)\right) f(t_{n-1})\, dt_{n-1}\ldots dt_1\end{aligned} \tag{41}$$

Proof of Main Theorem 16 (cont.)

First, writing (40) in Fourier representation, and then substituting Fourier transform of (41) into (40) with $f(t,x) = e^{i(kx-k^2t)}$, we obtain

$$(iG_+^0V - iG_-^0V)(-iG_-^0V)^{n-1}f(t)$$
$$= \frac{(-i)^n}{(2\pi)^n}\iiint e^{i(\xi x-\xi^2 t)}\left(\prod_{j=0}^{n-1} e^{-it_j(\xi_{j-1}^2-\xi_j^2)}\widehat{V}(t_j,\xi_{j-1}-\xi_j)\right)\, d^n\xi\; d^nt\; d\xi \tag{42}$$

where integration goes over $(t_0,\ldots,t_{n-1})$ with $-\infty < t_{n-1} \le t_{n-2} \le \ldots \le t_0 < \infty$ and $t_0 = s$, and $(\xi_{-1},\ldots,\xi_{n-1}) \in \mathbb{R}^n$, with $\xi_{-1} = \xi$, $\xi_0 = \eta$ and $\xi_{n-1} = k$.

Proof of Main Theorem 16 (cont.)

Thus, substituting (42) into (39), we obtain

$$S\left(e^{i(kx-k^2t)}\right) = e^{i(kx-k^2t)} + \sum_{n=1}^{\infty} \int e^{i(\xi x-\xi^2 t)} S_n'(\xi,k)\,d\xi$$

where

$$S_n'(\xi,k) = \frac{(-i)^n}{(2\pi)^n} \iint \left(\prod_{j=0}^{n-1} e^{it_j(\xi_{j-1}^2-\xi_j^2)} \widehat{V}(t_j,\xi_{j-1}-\xi_j)\right) d^n t\, d^n\xi$$

where the t integration goes over $-\infty < t_{n-1} \leq t_{n-2} \leq \ldots \leq t_0 < \infty$ and the ξ integration goes over $(\xi_0,\ldots,\xi_{n-1}) \in \mathbb{R}^n$, and $\xi_{-1} = \xi$ and $\xi_{n-1} = k$. □

Corollary 17

Let $\widehat{V} \in L^1(\mathbb{R}^2)$. *By Theorem 16, it follows that*

$$\mathsf{S} = I + \mathcal{S}'$$

where

$$S'(k, k+l) \equiv \sum_{n=1}^{\infty} S'_n(k, k+l) = -i(2\pi)^{-2} \int e^{il(l+2k)\eta}\, \widehat{V} * \widehat{\mu}^l(l, \eta; k)\; d\eta \tag{43}$$

is the same as (12) *up to* $\widehat{\mu}^r$ *being replaced by* $\widehat{\mu}^l$, *where* S'_n *are given by* (34)

Proof of Corollary 17.

Consider (12) with $\widehat{\mu}^r$ replaced by $\widehat{\mu}^l$.

Substitute $\widehat{\mu}^l(\xi, t; k) = e^{ik^2 t}\widehat{\psi}^l(\xi + k, t; k)$ with $\xi = k + l, t = \eta$ into the modified (12) using the series expansion of $\widehat{\psi}^l$,

$$S(k, k+l) = -\frac{i}{(2\pi)^2}\sum_{n=1}^{\infty}\iint \widehat{V}(\eta, l - \tilde{l})e^{ik^2\eta}\widehat{\psi}^l_{n-1}(k + \tilde{l}, \eta; k)e^{il(l+2k)\eta}\, d\tilde{l}\, d\eta \qquad (44)$$

where $\widehat{\psi}^l_n(\xi, t; k)$ are given by (36) and $\widehat{\psi}^l_0(\xi, t; k) = 2\pi\delta(\xi - k)e^{-ik^2 t}$.

Finally, substitute $\widehat{\psi}^l_{n-1}(\xi, t; k)$ into (44) with $\xi = k + l, t = \eta$,

$$S(k, k+1) = S'(k, k+l)$$

□

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Appendix A.1: Existence of Wave Operators on L^2

Proof of Theorem 1.(i).

Assumption 1 implies $V \in L^1(\mathbb{R}; L^\infty(\mathbb{R}^d))$. Let $\psi_0 \in L^2(\mathbb{R}^d)$. Then there exists a unique propagator $U(t,s)$ such that $\psi(t) = U(t,s)\psi_0$ simultaneously solves the Schrödinger equation and

$$\psi(t) = U_0(t-s)\psi_0 - i\int_s^t U_0(t-\tau)V(\tau)\psi(\tau)d\tau$$

Let $\varphi_0 \in L^2(\mathbb{R}^d)$ and let $W(s;t) = U(s,t)U_0(t-s)$. Then,

$$\begin{aligned} &< \psi_0, W(s;t)\varphi_0 > \\ &\qquad =< \psi_0, \varphi_0 > + i\int_s^t < \psi_0, U(s,\tau)V(\tau)U_0(\tau-s)\varphi_0 > d\tau \end{aligned}$$

So,

$$W(s;t)\varphi_0 = \varphi_0 + i\int_s^t U(s,\tau)V(\tau)U_0(\tau-s)\varphi_0 d\tau \qquad (45)$$

Appendix A.1: Existence of Wave Operators on L^2 (cont.)

Proof of Theorem 1.(i) (cont.)

Let $f(\tau, s) = U(s,\tau)V(\tau)U_0(\tau - s)\varphi_0$ Then

$$\|f(\tau,s)\|_2 \le \|V(\tau)\|_{L_x^\infty}\|\varphi_0\|_2$$

Also, let $W_+(s)\varphi_0 = \phi_0 + \int_s^\infty f(\tau,s)d\tau$. Then,

$$\|W_+(s)\varphi_0\|_2 \le \|\varphi_0\|_2 + \|V\|_{L_t^1(L_x^\infty)}\|\varphi_0\|_2$$

Now, we show $W_+(s) = st\text{ - }\lim_{t\to\infty} W(s;t)$.

$$\begin{aligned}\|W(s;t)\varphi_0 - W_+(s)\varphi_0\|_2 &\le \int_t^\infty \|f(\tau,s)\|_2 d\tau \\ &\le \|\varphi_0\|_2 \int_t^\infty \|V(\tau)\|_{L_x^\infty} d\tau \\ &\to 0 \quad \text{as} \quad t\to\infty\end{aligned}$$

since $V \in L_t^1(\mathbb{R}; L_x^\infty(\mathbb{R})^d)$.

Appendix A.1: Existence of Wave Operators on L^2 (cont.)

Proof of Theorem 1.(i) (cont.)

Next, we show $W_\pm(s)$ are unitary.
Let $Z(s;t) = U_0(s-t)U(t,s)$. In $\mathcal{B}(L^2)$, $W(s;t)^* = Z(s;t)$. Similar estimates can be done to show that

$$Z_\pm(s) = st\text{ - }\lim_{t\to\pm\infty} Z(s;t)$$

Then for $\varphi \in L^2(\mathbb{R}^d)$,

$$\begin{aligned} &\|W_+(s)Z_+(s)\varphi - W(s;t)Z(s;t)\varphi\|_2 \\ &\leq \|[W_+(s) - W(s;t)]Z_+(s)\varphi\|_2 + \|W(s;t)\|\|[Z_+(s) - Z(s;t)]\varphi\|_2 \\ &\to 0 \quad \text{as} \quad t\to\infty \end{aligned}$$

Thus,

$$\begin{aligned} W_+(s)Z_+(s) &= st\text{ - }\lim_{t\to\pm\infty} U(s,t)U_0(t-s)U_0(s-t)U(t,s) \\ &= 1 \end{aligned}$$

□

Appendix A.2: Existence of Wave Operators on L^p

Proof of Theorem 1.(ii).

By (45), for any $\varphi_0 \in L^2(\mathbb{R}^d)$, we have

$$W(s;t)\varphi_0 = \varphi_0 + i\int_s^t U(s,\tau)V(\tau)U_0(\tau - s)\phi_0 d\tau$$
$$= \varphi_0 + i\int_s^t W(s;\tau)U_0(s-\tau)V(\tau)U_0(\tau - s)\phi_0 d\tau \qquad (46)$$

Let $\tilde{V}(s;t) = U_0(s-t)V(t)U_0(t-s)$ be defined on $L^1 \cap L^2$.
First, we show that $\tilde{V}(s;t)$ extends to a bounded operator on $L^1(\mathbb{R}^d)$ such that for each $s \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \left\|\tilde{V}(s;t)\right\|_{\mathcal{B}(L^1)} dt \leq c\left\|\widehat{V}\right\|_{L^1(\mathbb{R};\mathcal{M}(\mathbb{R}^d))}$$

Appendix A.2: Existence of Wave Operators on L^p (cont.)

Proof of Theorem 1.(ii) (cont.)

Define a Fourier multiplier operator $U_0(t)$ on $L^1(\mathbb{R}^d)$ by

$$(U_0(t)\varphi)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi\cdot x} e^{-it|\xi|^2} \widehat{\varphi}(\xi) d\xi$$

Note that $(\mathcal{F}\{x_j\varphi\})(\xi) = i\frac{\partial}{\partial \xi_j}\widehat{\varphi}(\xi)$. Using integration by parts, it follows that

$$U_0(-t)xU_0(t) = x + 2tp = e^{-ix^2/4t} 2tp e^{ix^2/4t}$$

where $p = -i\nabla_x$.

Then as operators on $L^2(\mathbb{R}^d)$ we have with $t \neq s$,

$$\tilde{V}(s;t) = e^{-ix^2/4(t-s)} V(t, 2(t-s)p) e^{ix^2/4(t-s)}$$

Thus, $\left\|\tilde{V}(s;t)\varphi\right\|_{L^1(\mathbb{R}^d)} = \|V(t, 2(t-s)p)\varphi\|_{L^1(\mathbb{R}^d)}$.

Appendix A.2: Existence of Wave Operators on L^p (cont.)

Proof of Theorem 1.(ii) (cont.)

Define Fourier multiplier operators $V(t, 2(t-s)p)$ and $V(t,p)$ on $L^1(\mathbb{R}^d)$, respectively, by

$$(V(t,2(t-s)p)\varphi)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi\cdot x} V(t, 2(t-s)\xi)\hat{\varphi}(\xi) d\xi$$

$$(V(t,p)\varphi)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi\cdot x} V(t,\xi)\hat{\varphi}(\xi) d\xi \qquad (47)$$

By change of variables, it follows that

$$\|V(t,2(t-s)p)\varphi\|_{L^1(\mathbb{R}^d)} = \|V(t,p)\varphi\|_{L^1(\mathbb{R}^d)}$$

Appendix A.2: Existence of Wave Operators on L^p (cont.)

Proof of Theorem 1.(ii) (cont.)

Assumption 1 implies that

$$V(t,x) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} d\mu_t(\xi) \tag{48}$$

where μ_t is a complex Borel measure for fixed $t \in \mathbb{R}$. Using (47) and (48) with Fubini's theorem, we obtain for $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$(V(t,p)\varphi)(x) = \int_{\mathbb{R}} \varphi(x - x')d\mu_t(x')$$

Thus, using the polar decomposition of μ_t

$$\|V(t,p)\varphi\|_{L^1(\mathbb{R}^d)} \leq \|\varphi\|_{L^1(\mathbb{R}^d)}\|\mu_t\|_{\mathcal{M}(\mathbb{R}^d)}$$

where $\|\mu_t\|_{\mathcal{M}(\mathbb{R}^d)} = |\mu_t|(\mathbb{R}^d)$ is the total variation of of μ_t with $|\mu_t|$ being a positive Borel measure.

Appendix A.2: Existence of Wave Operators on L^p (cont.)

Proof of Theorem 1.(ii) (cont.)

Hence, we obtain

$$\begin{aligned}\left\|\tilde{V}(s;t)\right\|_{\mathcal{B}(L^1)} &= \|V(t, 2(t-s)p)\|_{\mathcal{B}(L^1)} \\ &= \|V(t,p)\|_{\mathcal{B}(L^1)} \\ &\leq \left\|\widehat{V}(t,\cdot)\right\|_{\mathcal{M}(\mathbb{R}^d)}\end{aligned}$$

Thus, $\tilde{V}(s;t)$ extends to a bounded operator on $L^1(\mathbb{R}^d)$ such that for each $s \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \|\tilde{V}(s;t)\|_{\mathcal{B}(L^1)} dt \leq \left\|\widehat{V}\right\|_{L^1(\mathbb{R};\mathcal{M}(\mathbb{R}^d))}$$

Appendix A.2: Existence of Wave Operators on L^p (cont.)

Proof of Theorem 1.(ii) (cont.)

By (46), we have

$$W(s;t)\varphi_0 = \varphi_0 + i\int_s^t W(s;\tau)\tilde{V}(s;\tau)\varphi_0 d\tau \tag{49}$$

A Dyson series for the solution of (46)

$$W(s;t)\varphi_0 = \varphi_0 + \sum_{n\geq 1} W^{(n)}(s;t)\varphi_0$$

where the nth term in the series is given by

$$W^{(n)}(s;t)\varphi_0$$
$$= i^n \int_{s\leq t_1\leq\cdots\leq t_n\leq t} \prod_{k=1}^{n} \tilde{V}(s;t_k)\varphi_0 dt_1\cdots dt_n$$

Appendix A.2: Existence of Wave Operators on L^p (cont.)

Proof of Theorem 1.(ii) (cont.)

Let $W_+(s)\varphi_0 = \varphi_0 + \sum_{n\geq 1} W_+^{(n)}(s)\phi_0$ with

$$W_+{}^{(n)}(s)\varphi_0 = i^n \int_{s\leq t_1\leq\cdots\leq t_n\leq\infty} \prod_{k=1}^{n} \tilde{V}(s;t_k)\varphi_0 dt_1\cdots dt_n$$

Then,

$$\begin{aligned}\left\|W_+^{(n)}(s)\varphi_0\right\|_{L^1} &= \int_{s\leq t_1\leq\cdots\leq t_n\leq\infty} \prod_{k=1}^{n} \left\|\tilde{V}(s;t_k)\right\|_{\mathcal{B}(L^1)}\left\|\varphi_0\right\|_{L^1} dt_1\cdots dt_n \\ &= \frac{1}{n!}\left[\int_s^\infty \left\|\tilde{V}(s;\tau)\right\|_{\mathcal{B}(L^1)} d\tau\right]^n \left\|\varphi_0\right\|_{L^1} \\ &\leq \frac{1}{n!}\left[\left\|\widehat{V}\right\|_{L^1(\mathbb{R};\mathcal{M}(\mathbb{R}))}\right]^n \left\|\varphi_0\right\|_{L^1}\end{aligned}$$

Appendix A.2: Existence of Wave Operators on L^p (cont.)

Proof of Theorem 1.(ii) (cont.)

Thus,

$$\|W_+\varphi_0\|_{L^1} \leq \exp\left[\left\|\widehat{V}\right\|_{L^1(\mathbb{R};\mathcal{M}(\mathbb{R}^d))}\right]\|\varphi_0\|_{L^1}$$

Similarly, we obtain

$$\|W(s;t)\varphi_0 - W_+(s)\varphi_0\|_{L^1} \leq \left[e^{\int_t^\infty \|\tilde{V}(s;\tau)\|_{\mathcal{B}(L^1)}d\tau} - 1\right]\|\varphi_0\|_{L^1}$$

$$\to 0 \quad \text{as} \quad t \to \infty$$

Thus, $W_\pm(s)$ extend to a bounded operators on $L^1(\mathbb{R}^d)$.
Similar estimate can be done to show that $W_\pm(s)^*$ also extends to bounded operators on $L^1(R^d)$.
By duality, $W_\pm(s)$ extend to bounded operators on $L^\infty(\mathbb{R}^d)$.
By interpolation, $W_\pm(s)$ extend to bounded operators on $L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$. □

$G^0_\pm$, $G_\pm$ are bounded from $L(Q)$ to $L(P)$ (cont.)

Proof of Lemma 3.(i).

First, we show that if $q \geq 2$ and $t \neq 0$,

$$\|U_0(t)\varphi\|_q \leq c|t|^{-d(1/2-1/p)}\|\varphi\|_{q'}$$

. Let $E(t) = e^{ix^2/4t}$. Then,

$$\begin{aligned} U_0(t)\varphi_{(x)} &= (4\pi it)^{-d/2}\int_{\mathbb{R}^d} e^{i|x-y|^2/4t}\varphi(y)dy \\ &= (4\pi it)^{-d/2}E(t)t^{-d/2}(\mathcal{F}E(t)\varphi)(x/2t) \end{aligned}$$

So,

$$\begin{aligned} \|U_0(t)\varphi\|_q &= (4\pi)^{-d/2}\|\mathcal{F}E(t)\varphi(\cdot/2t)\|_q \\ &= c|t|^{-d(1/2-1/q)}\|\mathcal{F}E(t)\varphi\|_q \\ &\leq c|t|^{-d(1/2-1/q)}\|\varphi\|_{q'} \end{aligned}$$

$G^0_\pm$, $G_\pm$ are bounded from $L(Q)$ to $L(P)$ (cont.)

Proof of Lemma 3.(i) (cont.)

Let $P = (1/q, 1/r) \in [BC[$ and $P' = (1/q', 1/r') \in [B'C'[$.
It follows that

$$\left\| G^0_\pm \varphi \right\|_q \leq c \int_{\pm\infty}^{t} |t-s|^{-2/r} \|\varphi(s)\|_{q'} ds$$

Since $1/r' - 1/r = 1 - 2/r > 0$, by Hardy-Littlewood-Sobolev inequality

$$\left\| G^0_\pm \varphi \right\|_{L(P)} \leq c \|\varphi\|_{L(P')}$$

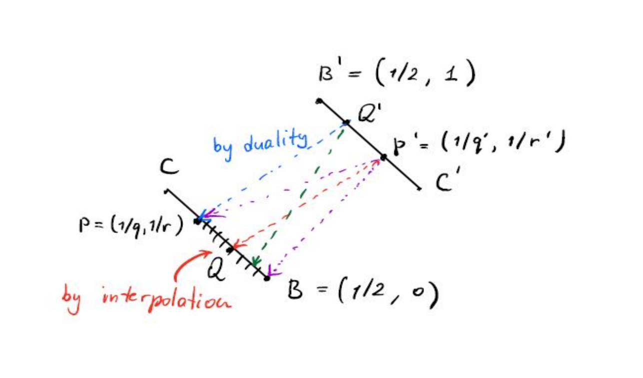
Assume $\varphi(t) \in L^2 \cap L^{q'}$.

$$\left\| G^0_\pm \varphi(t) \right\|_2^2 = \int_{\pm\infty}^{t} \int_{\pm\infty}^{t} < \phi(s), U_0(s-s')\varphi(s') > ds\, ds'$$

$$\leq 2\,\mathrm{Re} \int_{\pm\infty}^{t} < \varphi(s), G^0_\pm \varphi(s) > ds \leq c \|\varphi\|^2_{L(P')}$$

$G_{\pm}^{0}$, $G_{\pm}$ are bounded from $L(Q)$ to $L(P)$ (cont.)

Proof of Lemma 3.(i) (cont.)



Thus, $G_{\pm}^{0}$ is bounded from any $L(\bar{P})$ with $\bar{P} \in [B'C'[$ to any $L(P)$ with $P \in [BC[$.

$G_{\pm}^{0}$, $G_{\pm}$ are bounded from $L(Q)$ to $L(P)$ (cont.)

Proof of Lemma 3.(ii).

Let $P = (1/q, 1/r) \in T$ and $\bar{P} = (1/\bar{q}, 1/\bar{r}) \in T'$ with $\dfrac{1}{q} + \dfrac{1}{\bar{q}} = 1$ and $\dfrac{1}{\bar{q}} + \dfrac{2}{\bar{r}d} - \dfrac{1}{q} - \dfrac{2}{rd} = \dfrac{2}{d}$. So, $\dfrac{1}{\bar{r}} - \dfrac{1}{r} = 1 - d\left(\dfrac{1}{2} - \dfrac{1}{q}\right)$ and we have

$$\left\| G_{\pm}^{0} \varphi \right\|_{q} \leq c \int_{\pm\infty}^{t} |t-s|^{-d(1/2-1/q)} \|\varphi(s)\|_{\bar{q}} ds$$

Then, by Hardy-Littlewood-Sobolev inequality, $G_{\pm}^{0}$ is bounded from $L(\bar{P})$ to $L(P)$.

$G^0_\pm$, $G_\pm$ are bounded from $L(Q)$ to $L(P)$ (cont.)

Lemma 18 (Interpolation Lemma)

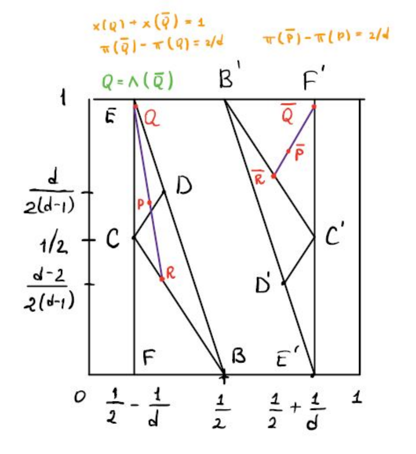
Assume that none of P, $\bar{P}$, Q, $\bar{Q}$ has height zero. If a linear operator maps $L(\bar{P})$ into $L(P)$ and $L(\bar{Q})$ into $L(Q$ (continuously), then it maps $L((1-\theta)\bar{P}+\theta\bar{Q})$ into $L((1-\theta)P+\theta Q)$, where $0<\theta<1$.

Proof of Lemma 3 (cont.)

Consider the map $\bar{P} \to P$ in Lemma 10.(ii) with $x(P)+x(\bar{P})=1$ and $\pi(\bar{P})-\pi(P)=2/d$. Extend this map to an affine map Λ of $cl(T')$ onto $cl(T)$. Note that $\Lambda(B')=B$, $\Lambda(E')=F$ and $\Lambda(F')=E$.
Take any pair $P \in T$ and $\bar{P} \in T'$ with $\pi(\bar{P})$ to $L(P)$. We show that $G^0_\pm$ maps $L(\bar{P})$ to $L(P)$.

$G^0_\pm$, $G_\pm$ are bounded from $L(Q)$ to $L(P)$ (cont.)

Proof of Lemma 3 (cont.)



$G^0_\pm$, $G_\pm$ are bounded from $L(Q)$ to $L(P)$ (cont.)

Proof of Lemma 3 (cont.)

$G^0_\pm$ map $L(\bar{Q})$ to $L(Q)$ by (ii) and $L(\bar{R})$ to $L(R)$ by (i). If we show that P divided $[QR]$ at the same ratio as $\bar{P}$ does $[\bar{Q}\bar{R}]$, then using the Interpolation Lemma, we complete the proof of Theorem 2.

Choose t such that $\bar{P} = (1-t)\bar{Q} + t\bar{R}$. Then,

$$\pi(\bar{P}) = (1-t)\pi(\bar{Q}) + t\pi(\bar{R})$$

On the other hand, we have $\pi(\bar{Q}) = \pi(Q) + 2/d$ and $\pi(\bar{P}) = \pi(P) + 2/d$. Then,

$$\pi(P) = \pi(\bar{P}) - 2/d = \pi((1-t)Q + tR)$$

Since π is injective on $[QR]$, which has slop different from $-d/2$,

$$P = (1-t)Q + tR$$

as required. For $\bar{P}$ to be below $[B'C'[$, choose $\bar{Q}$ close to E' and repeat the above arguments. □